Hold on to your homework for now...

Remember:
- Friday is SRS in Tucker Building
- Thurs May 4 — Calculus Bee, \$100, \$75, \$50 prizes,

  Refreshments 3:30PM / Competition 4 PM. (Star wars theme) } TUC 244

Using Taylor series to make "interesting" calculations:

Ex $\frac{5^3}{2!} - \frac{5^4}{3!} + \frac{5^5}{4!} - \frac{5^6}{5!} + \cdots = \boxed{?}$

Solution: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges to e^x for all x.

$$\Rightarrow e^{-5} = 1 + -5 + \frac{(-5)^2}{2!} + \frac{(-5)^3}{3!} + \frac{(-5)^4}{4!} + \cdots$$

$$= \underbrace{1 - 5}_{-4} + \frac{5^2}{2!} - \frac{5^3}{3!} + \frac{5^4}{4!} - \frac{5^5}{5!} + \cdots$$

$$\Rightarrow 5e^{-5} = -20 + \frac{5^3}{2!} - \frac{5^4}{3!} + \frac{5^5}{4!} - \frac{5^6}{5!} + \cdots$$

$$\Leftrightarrow \boxed{5e^{-5} + 20} = \frac{5^3}{2!} - \frac{5^4}{3!} + \frac{5^5}{4!} - \frac{5^6}{5!} + \cdots$$

Hmk
- $5+5+\frac{5}{2!}+\frac{5}{3!}+\frac{5}{4!}\cdots$

$e^1 = 1 + \underset{x=1}{1} + \underset{x^2=1}{\frac{1}{2!}} + \underset{x^3=1}{\frac{1}{3!}} + \frac{1}{4!} + \cdots$

$5e$

- $1-\frac{4}{2!}+\frac{16}{4!}-\frac{64}{6!}+\cdots$

$\cos(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

$x=2$

- $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\frac{1}{7\cdot 8}+\frac{1}{9\cdot 10}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

partial fractions

$= \frac{2-1}{1\cdot 2}+\frac{4-3}{3\cdot 4}+\frac{6-5}{5\cdot 6}+\cdots$

$= \frac{2}{1\cdot 2}-\frac{1}{1\cdot 2}+\frac{4}{3\cdot 4}-\frac{3}{3\cdot 4}+\frac{6}{5\cdot 6}-\frac{5}{5\cdot 6}+\cdots$

$= \frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots$

alternating harmonic series.

It would be convenient if we had a Taylor series

$F(x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots$

(Then plug in $x=1$)

$F'(x) = 1-x+x^2-x^3+x^4-x^5+\cdots$

geometric $a=1$, $r=-x$

$F'(x) = \frac{1}{1-(-x)} = \frac{1}{1+x}$

$\Rightarrow F(x) = \ln|1+x| + C$

$$\longrightarrow \quad F(0) = \ln|1+0| + C$$

$$0 = 0 + C \implies C = 0$$

$$F(x) = \ln(1+x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

$\Rightarrow x = 1$

$$F(1) = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$= \boxed{\ln 2}.$$

$\boxed{Ex}$ Last time

$$\arctan(x) = \int_0^x \underbrace{\frac{1}{1+x^2}}_{\text{geometric } a=1,\ r=-x^2} dx$$

$$= \int_0^x \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

true if x is between -1 & 1.

Converges if $x \in (-1, 1]$

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

nifty formula for π

$$\Rightarrow \boxed{\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots}$$

The interval of convergence of a Taylor series $\sum_{k=0}^{\infty} a_k (x-a)^k$ is the set of x's for which the series converges.

To find the radius of convergence, use ratio or root tests $\longrightarrow$ set $L < 1$.

It turns out that the interval of convergence is always one of these forms.

① $(-\infty, \infty)$ converges for all x (e^x, $\cos(x)$, $\sin(x)$).

② $(a-r, a+r)$

e.g. geometric

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

converges for $-1 < x < 1$

interval $(-1, 1)$.

③ ⓐ $[a-r, a+r)$

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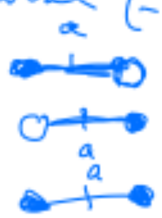


$\boxed{Ex}$ Find interval of convergence of the Taylor series of arctan(x) around x=0.

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

last time (above also)

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2(k+1)}}{2k+1}.$$

$\left(\begin{array}{c}\text{Ratio test}\\ \text{Limit}\end{array}\right) < 1$ → give me most of the interval of convergence.

Then we'll check the endpoints.

$$L = \text{ratio limit} = \lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| < 1$$

$$b_k = \frac{(-1)^k x^{2k+1}}{2k+1} \Rightarrow b_{k+1} = \frac{(-1)^{(k+1)} x^{2(k+1)+1}}{2(k+1)+1}$$

$$\Rightarrow b_{k+1} = \frac{(-1)^{k+1} x^{2k+3}}{2k+3}$$

$$\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1} x^{2k+3}}{2k+3}\right| \cdot \left|\frac{2k+1}{(-1)^k x^{2k+1}}\right| < 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{|x|^{2k+3}}{(2k+3)} \cdot \frac{(2k+1)}{|x|^{2k+1}} < 1$$

$$\Rightarrow \lim_{k\to\infty} \frac{(2k+1)}{(2k+3)} \frac{|x|^{2k+3}}{|x|^{2k+1}} < 1$$

$$\Rightarrow \lim_{k\to\infty}\left|\frac{2k+1}{2k+3}\right| \cdot |x|^2 < 1$$

$$\left(\frac{1+\frac{1}{2k}}{1+\frac{3}{2k}}\right) \to 1$$

$$\Rightarrow |x|^2 < 1$$

$$|x| < 1$$

$$-1 < x < 1$$

radius of convergence is 1.

(divergence $|x| > 1$) Question remains — does it converge at $x=1$? at $x=-1$?

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\overset{x=1}{=} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

Converges by alternating series test

- $\frac{1}{2k+1}$ decreasing
- $\frac{1}{2k+1} \to 0$.
- alternating

Converges at $x=1$.

$$\overset{x=-1}{=} -1 - \frac{(-1)^3}{3} + \frac{(-1)^5}{5} - \frac{(-1)^7}{7} + \cdots$$

$$= -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots$$

Converges by alternating series test!

Converges at $x=-1$ also.

$\therefore$ interval of convergence is $[-1, 1]$.

Ex Find the Taylor series of $\frac{1}{x}$ around $x=-2$, and find its interval of convergence.

Solution ①

$$f(x) = x^{-1} \overset{x=-2}{=} (-2)^{-1} = \frac{1}{(-2)}$$

$$f'(x) = (-1)x^{-2} = (-1)(-2)^{-2} = \frac{1}{-2^2}$$

$$f''(x) = (-1)(-2)x^{-3} = (-1)(-2)(-2)^{-3} = -\frac{2!}{2^3}$$

$$f'''(x) = (-1)(-2)(-3)x^{-4} = (-1)(-2)(-3)(-2)^{-4} = -\frac{3!}{2^4}$$

$$\vdots$$

$$f^{(n)}(x)\Big|_{(x=-2)} = -\frac{n!}{2^{n+1}}$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} -\frac{n!}{2^{(n+1)}} \frac{1}{n!}(x+2)^n$$

$$= \sum_{n=0}^{\infty} -\frac{(x+2)^n}{2^{n+1}}$$

Another method:

$$\frac{1}{x} = \frac{1}{x+2-2} = -\frac{1}{2}\frac{1}{\left(1-\left(\frac{x+2}{2}\right)\right)}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x+2}{2}\right)^n = \sum_{n=0}^{\infty} -\frac{(x+2)^n}{2^{n+1}}$$

Ratio Test

$$b_k = -\frac{(x+2)^k}{2^{k+1}} \quad b_{k+1} = \frac{-(x+2)^{k+1}}{2^{k+2}}$$

$$L = \lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty} \left| \frac{-(x+2)^{k+1}}{2^{k+2}} \cdot \frac{-(2^{k+1})}{(x+2)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(x+2)^{k+1}}{(x+2)^k} \cdot \frac{2^{k+1}}{2^{k+2}} \right| < 1$$

$$\Rightarrow \lim_{k\to\infty} |x+2| \cdot \frac{1}{2} < 1$$

$$\frac{|x+2|}{2} < 1 \Rightarrow |x+2| < 2$$

$$-2 < x+2 < 2$$

$$-4 < x < 0$$

radius of convergence is 2

definitely converges for $-4 < x < 0$.
diverges for $x > 0$, $x < -4$.

Question remains: what about $x=0$, $x=-4$?

$$\sum_{n=0}^{\infty} -\frac{(x+2)^n}{2^{n+1}} \overset{x=-4}{=} \sum_{n=0}^{\infty} -\frac{(-4+2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} -\frac{(-2)^n}{2^{n+1}}$$

$-2 = (-1)2$

$$= \sum_{n=0}^{\infty} -\frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2} = -\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \cdots$$

diverges (terms don't go to 0).

diverges @ $x=-4$.

$$\sum_{n=0}^{\infty} -\frac{(x+2)^n}{2^{n+1}} \overset{x=0}{=} \sum_{n=0}^{\infty} -\frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} -\frac{1}{2}$$

$$= -\frac{1}{2} + -\frac{1}{2} + -\frac{1}{2} + -\frac{1}{2} + \cdots = -\infty.$$

diverges.

$\therefore$ diverges @ $x=0$.

Interval of convergence is $\boxed{(-4,0)}$

$\boxed{\text{Example}}$ Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(n!)^2}{n^n} x^n$.

Ratio test: $b_n = \dfrac{(n+1)(n!)^2}{n^n} x^n$, $b_{n+1} = \dfrac{(n+2)((n+1)!)^2}{(n+1)^{n+1}} x^{n+1}$

$$C = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{(n+2)((n+1)!)^2}{(n+1)^{n+1}} x^{n+1} \cdot \frac{n^n}{(n+1)(n!)^2 x^n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+2)}{(n+1)} \xrightarrow{} 1 \; \frac{((n+1)!)^2}{(n!)^2} \frac{n^n}{(n+1)^{n+1}} \frac{|x|^{n+1}}{|x|^n}$$

$$= \lim_{n\to\infty} 1 \cdot (n+1)^2 \frac{\cancel{(n!)^2}}{\cancel{(n!)^2}} \cdot \frac{n^n}{(n+1)^n (n+1)} |x|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)} \cdot \left(\frac{n}{n+1}\right)^n \right| |x| < 1$$

$$\left(\frac{n+1}{n}\right)^n \to e$$

$$\left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$$

$$= \lim_{n\to\infty} (n+1) \cdot \frac{1}{e} |x| < 1$$

$$= \infty \text{ unless } |x| = 0$$

So the only x for which the series converges is $x = 0$!

set of convergence $= \{0\}$
$= [0,0]$.

Absolute & Conditional Convergence of a series.

We say $\sum_{k=0}^{\infty} a_k$ converges absolutely if $\sum_{k=0}^{\infty} |a_k|$ converges.

Lemma: If a series converges absolutely, then it converges.

But the converse is false. It's possible that $\sum a_k$ converges but $\sum |a_k|$ diverges.

When this happens, we say that $\sum a_k$ converges <u>conditionally</u>.

<u>Example</u> Alternating Harmonic Series

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$ converges to $\ln(2)$.

but $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ diverges (to $+\infty$)

— harmonic series

$\sum_{k=1}^{\infty} \frac{1}{k}$. ie $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ <u>converges conditionally</u>.

<u>Riemann Rearrangement Theorem</u>.

- If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then after rearranging, you can make it add to any number (including $\pm\infty$).
- If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement results in the same sum.

Example $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots = \ln(2)$.

but $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots > \ln(2)$.

We can also rearrange the terms so that

$$1+\frac{1}{3}+1\ldots\ldots - \cdot - = \pi.$$

Parametrized Curves & Polar Coordinates.

Graphs – $y=x^2$

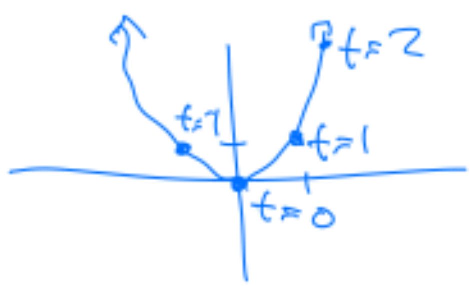
$x^2+y^2=4$

Implicit equations of curves

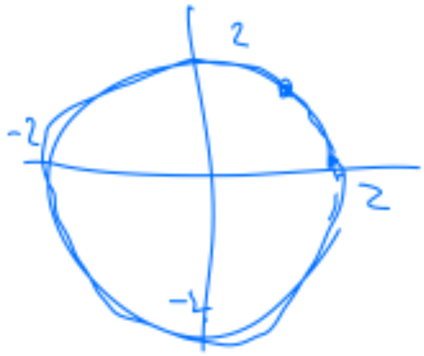
parametrised equations – x & y depend on t (time)

$x(t)$
$y(t)$

$x(t)=t$
$y(t)=t^2$
$(x,y)=(t,t^2)$

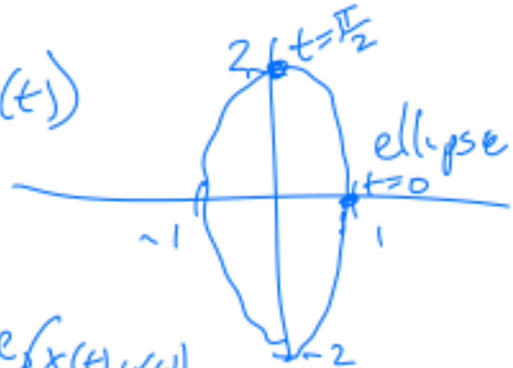


$x(t) = 2\cos(t)$
$y(t) = 2\sin(t)$
$0 \le t \le 2\pi$



$(x,y) = (2\cos(t), 2\sin(t))$
for $0 \le t \le 2\pi$.

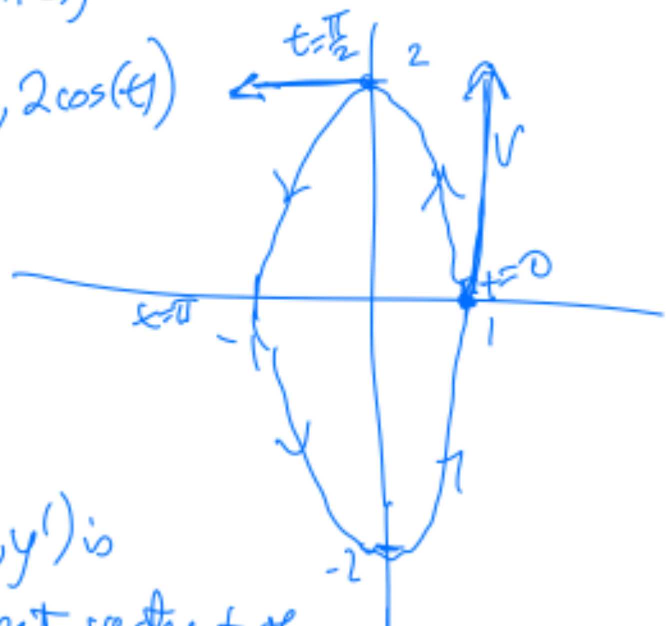
$(x, y) = (\cos(t), 2\sin(t))$
$0 \le t \le 2\pi$



Given a parametrized curve $(x(t), y(t))$
it's velocity vector is $v = (x'(t), y'(t))$

$(x,y) = (\cos(t), 2\sin(t))$
$v = (x', y') = (-\sin(t), 2\cos(t))$

$t = 0$
$v = (0, 2)$
$t = \frac{\pi}{2}$
$v = (-1, 0)$



The velocity vector $v = (x', y')$ is
also called the tangent vector to the curve.
The length of the velocity vector is the speed

$$V = (x', y')$$

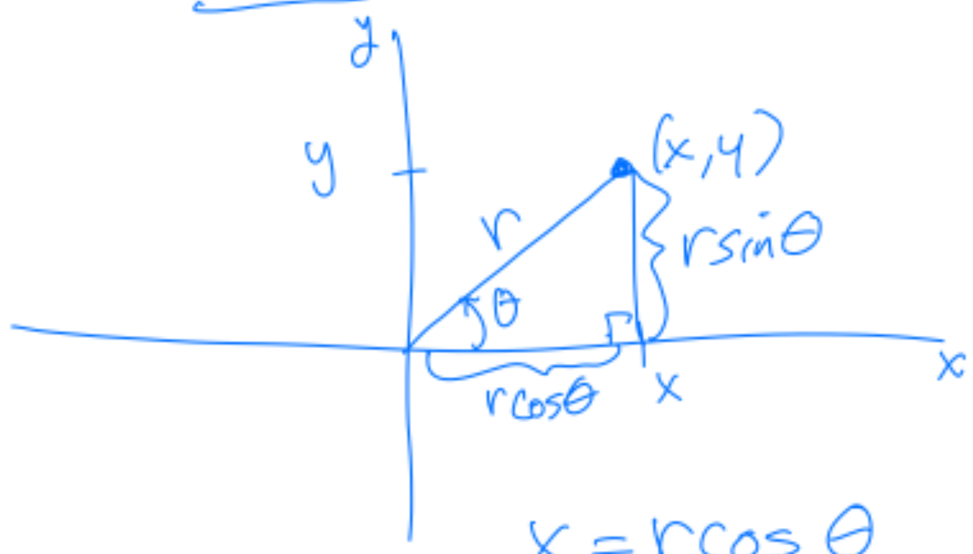
length of V $= |V| = |(x', y')| = \sqrt{(x')^2 + (y')^2}$.

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

To find the length of a curve $\alpha(t) = (x(t), y(t))$ from $t=a$ to $t=b$.

$$L(\alpha) = \int_a^b \underbrace{\sqrt{(x')^2 + (y')^2}}_{\text{speed}} \, \underbrace{dt}_{\text{time}}$$

Polar coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta.$$